— Exercises —

1. **Borel realization theorem.** Let $(a_n)_{n\geq 0}$ be an arbitrary sequence of numbers. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function s.t. $\varphi(x) = 1$ if $x \in [-1, 1]$ and $\varphi(x) = 0$ if $|x| > 2$. Define $\forall n \geq 0, \varphi_n = x^n \varphi(x)$ and denote by $M_n$ an upper bound of $|\varphi_n|, |\varphi_n'|, \ldots, |\varphi_n^{(n-1)}|$ on $\mathbb{R}$. Choose a sequence $(\lambda_n)_{n\geq 0}$ tending to $+\infty$, s.t. $\forall n, \lambda_n \geq 1$ and $\sum_{n\geq 0} |a_n| M_n/\lambda_n$ converges. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sum_{n\geq 0} a_n x^n \varphi(\lambda_n x)$.

   (a) Check that everything is well-defined.

   (b) Show by induction on $p$ that $\forall p \geq 0$, $f$ is $\mathcal{C}^p$ and $\forall x, f^{(p)}(x) = \sum_{n\geq 0} a_n \lambda_n^{p-n} \varphi_n^{(p)}(\lambda_n x)$.
   *Hint: at each induction step, show the normal convergence of the series of derivatives.*

   (c) Conclude that $\forall p \geq 0, f^{(p)}(0)/p! = a_p$.

— Problems —

2. **Solving differential equations using power series.** Text p. 583 12.59

3.* **Weak Tauberian Theorem** Let $\sum_{n\geq 0} a_n x^n$ be a power series with radius of convergence 1. Let $f : (-1, 1) \to \mathbb{R}$ be its sum. Assume that $f(x) \to S$ when $x \xrightarrow{x<1} 1$ and $na_n \xrightarrow{n\to\infty} 0$. Show that $\sum_{n\geq 0} a_n$ converges to $S$.

4. **Riemann-Lebesgue lemma.** Let $\varphi : \mathbb{R} \to \mathbb{C}$ be a $2\pi$-periodic piecewise continuous function. Let $f : [a, b] \to \mathbb{C}$ be a piecewise continuous function. We want to show that

$$\lim_{n\to+\infty} \int_a^b f(t) e^{int} dt = 0.$$

   (a) Show that

$$\lim_{n\to+\infty} \int_a^b f(t)\varphi(nt)dt = \frac{1}{2\pi} \left( \int_0^{2\pi} \varphi(t)dt \right) \left( \int_a^b f(t)dt \right).$$

   *Hints: Denote $K = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t)dt$. We want to show that*

$$\lim_{n\to+\infty} \int_a^b f(t)\Big(\varphi(nt) - K\Big)dt = 0, \textit{ i.e.,}$$

   *$\lim_{n\to+\infty} \int_a^b f(t)\psi(nt)dt = 0$ with $\psi = \varphi - K$. Note that $\int_0^{2\pi} \psi(t)dt = 0$. Prove the result in three steps: first when $f$ is a characteristic function $\chi_{[\alpha,\beta]}$ with $[\alpha, \beta] \subset [a, b]$, then when $f$ is a step function, then for a general $f$, which can be approximated by a step function.*

   (b) Show that $\int_0^{2\pi} e^{it} dt = 0$ and conclude.